

# Chapter 1

## The Lovely but Lonely Vickrey Auction

Lawrence M. Ausubel and Paul Milgrom

### 1. *Introduction*

William Vickrey's (1961) inquiry into auctions and "counterspeculation" marked the first serious attempt by an economist to analyze the details of market rules and to design new rules to achieve superior performance. He demonstrated that a particular pricing rule makes it a dominant strategy for bidders to report their values truthfully, even when they know that their reported values will be used to allocate goods efficiently. Vickrey's discovery was largely ignored for a decade, but the floodgates have since opened. Dozens of studies have extended his design to new environments, developed his associated theory of bidding in auctions, and tested its implications using laboratory experiments and field data.

Despite the enthusiasm that the Vickrey mechanism and its extensions generate among economists, practical applications of Vickrey's design are rare at best. The classic English auction of Sotheby's and Christie's, in which bidders iteratively submit successively higher bids and the final bidder wins the item in return for a payment equaling his final bid, is closely related to Vickrey's second-price sealed-bid auction, but long predates it. Online auctions such as eBay, in which bidders commonly utilize proxy bids authorizing the auctioneer to bid up to specified prices on their behalf, more nearly resemble the Vickrey design for a single item; however, these remain true dynamic auctions, as online bidders who submit proxy bids generally retain the ability to raise

their proxy bids later. The most general and novel version of Vickrey's design, which applies to sales in which different bidders may want multiple units of homogeneous goods—or packages of heterogeneous goods—remains largely unused.

Why is the Vickrey auction design, which is so lovely in theory, so lonely in practice? The answer, we believe, is a cautionary tale that emphasizes the importance of analyzing practical designs from many perspectives. Vickrey's design has some impressive theoretical virtues, but it also suffers from weaknesses that are frequently decisive. This chapter reviews the theoretical pluses and minuses of the Vickrey design, highlighting issues that cannot be ignored in developing practical auction designs.

## **2. Description of the general Vickrey (VCG) design**

Vickrey's original inquiry treated both auctions of a single item and auctions of multiple identical items, providing a mechanism in which it is a dominant strategy for bidders to report their values truthfully and in which outcomes are efficient. For a single item, the mechanism is often referred to as the *second-price sealed-bid auction*, or simply the *Vickrey auction*. Bidders simultaneously submit sealed bids for the item. The highest bidder wins the item, but (unlike standard sealed-bid tenders) the winner pays the amount of the second-highest bid. For example, if the winning bidder bids 10 and the highest losing bid is 8, the winner pays 8. With these rules, a winning bidder can never affect the price it pays, so there is no incentive for any bidder to misrepresent his value. From bidder  $n$ 's perspective, the amount he bids determines only whether he wins, and only by bidding his true value can he be sure to win exactly when he is willing to pay the price.

In Vickrey's original treatment of multiple units of a homogeneous good, which may be available in either continuous or discrete quantities, each bidder is assumed to have

monotonically nonincreasing marginal values for the good. The bidders simultaneously submit sealed bids comprising demand curves. The seller combines the individual demand curves in the usual way to determine an aggregate demand curve and a clearing price for  $S$  units. Each bidder wins the quantity he demanded at the clearing price. However, rather than paying the prices he bid or the clearing price for his units, a winning bidder pays the *opportunity cost* for the units won.

In the case of discrete units, an equivalent way to describe the multi-unit Vickrey auction is that each bidder submits a number of separate bids, each representing an offer to buy one unit. These individual bids describe the bidder's demand curve. The auctioneer accepts the  $S$  highest bids. If bidder  $n$  wins  $K$  units, then he pays the sum of the  $K$  highest rejected bids by other bidders. For example, if a bidder wins 2 units and the highest rejected bids by his competitors are 12 and 11, then the bidder pays 23 for his two units.

Another way to describe the rule is that the price a bidder pays for his  $r^{\text{th}}$  unit is the clearing price that would have resulted if bidder  $n$  had restricted his demand to  $r$  units (all other bidders' behaviors held fixed). This equivalent description makes clear the opportunity-cost interpretation of the winners' payments. The total payment for bidder  $n$  is computed by summing this payment over all items won, in the case of discrete units, or by integrating this payment from 0 to the quantity won, in the case of continuous units.

The mechanism can be used as either a mechanism to sell (a standard auction) or as a mechanism to buy (a "reverse" auction). Described as a standard auction, the buyers generally pay a discount as compared to the clearing price. Described as a reverse auction, the sellers generally receive a premium as compared to the clearing price.

Indeed, the main point of Vickrey's seminal article was that the government cannot establish a marketing agency to implement a dominant-strategy mechanism in two-sided markets without providing a subsidy: "The basic drawback to this scheme is, of course, that the marketing agency will be required to make payments to suppliers in an amount that exceeds, in the aggregate, the receipts from purchasers" (Vickrey 1961, p. 13).

Since Vickrey's original contribution, his auction design has been melded with the Clarke-Groves design for public goods problems.<sup>1</sup> The resulting auction design works for heterogeneous goods as well as homogeneous goods and does not require that bidders have nonincreasing marginal values. As with Vickrey's original design, this mechanism still assigns goods efficiently and still charges bidders the opportunity cost of the items they win. The main difference is that the amounts paid cannot generally be expressed as the sums of bids for individual items. The extended Vickrey mechanism goes by various names. Here, we call it the *Vickrey-Clarke-Groves* or *VCG mechanism*.

Formally, the VCG mechanism is described as follows. Let  $\bar{x}$  be a vector of goods that a seller has on offer and let  $v_n(x_n)$  denote bidder  $n$ 's value for any nonnegative vector  $x_n$ . Each bidder  $n = 1, \dots, N$  reports a value function  $\hat{v}_n$  to the auctioneer. The auctioneer then computes a value-maximizing allocation:  $x^* \in \arg \max_{x_1, \dots, x_N} \sum_n \hat{v}_n(x_n)$  subject to  $\sum_n x_n \leq \bar{x}$ . The price paid by a bidder  $n$  is then  $p_n = \alpha_n - \sum_{m \neq n} \hat{v}_m(x_m^*)$ , where  $\alpha_n = \max \left\{ \sum_{m \neq n} \hat{v}_m(x_m) \mid \sum_{m \neq n} x_m \leq \bar{x} \right\}$ . Notice that  $\alpha_n$  depends only on the value reports of the other bidders and not on what bidder  $n$  reports.

To illustrate the VCG mechanism, suppose that there are two items for sale (A and B) and two bidders. Each bidder  $n = 1, 2$  submits bids:  $\hat{v}_n(A)$  for item A;  $\hat{v}_n(B)$  for item B; and  $\hat{v}_n(AB)$  for the two items together. Assume without loss of generality that  $\hat{v}_1(AB) \geq \hat{v}_2(AB)$  and  $\hat{v}_1(A) + \hat{v}_2(B) \geq \hat{v}_1(B) + \hat{v}_2(A)$ . If  $\hat{v}_1(AB) > \hat{v}_1(A) + \hat{v}_2(B)$ , then the outcome is that bidder 1 wins both items. Applying the formula, his payment is  $\hat{v}_2(AB)$ . However, if  $\hat{v}_1(AB) < \hat{v}_1(A) + \hat{v}_2(B)$ , then the outcome is that bidder 1 wins item A (with an associated payment of  $\hat{v}_2(AB) - \hat{v}_2(B)$ ) and bidder 2 wins item B (with an associated payment of  $\hat{v}_1(AB) - \hat{v}_1(A)$ ). In each case, the winner pays the opportunity cost of the items won, and his payment depends only on his opponent's reports.

The first theorem confirms that the general VCG mechanism still has the properties that it is a dominant strategy for each bidder to report its values truthfully and that the outcome in that event is an efficient one.

Theorem 1. Truthful reporting is a dominant strategy for each bidder in the VCG mechanism. Moreover, when each bidder reports truthfully, the outcome of the mechanism is one that maximizes total value.

Proof. Consider any fixed profile of reports,  $\{\hat{v}_m\}_{m \neq n}$ , for the bidders besides  $n$ . Suppose that when bidder  $n$  reports truthfully, the resulting allocation and payment vectors are denoted by  $x^*$  and  $p^*$ , but when bidder  $n$  reports  $\hat{v}_n$ , the resulting vectors are  $\hat{x}$  and  $\hat{p}$ . When bidder  $n$  reports  $\hat{v}_n$ , its payoff is:

$$\begin{aligned}
v_n(\hat{x}_n) - \hat{p}_n &= v_n(\hat{x}_n) + \sum_{m \neq n} \hat{v}_m(\hat{x}_m) - \alpha_n \\
&\leq \max \left\{ v_n(x_n) + \sum_{m \neq n} \hat{v}_m(x_m) \mid \sum_m x_m \leq \bar{x} \right\} - \alpha_n \\
&= v_n(x_n^*) + \sum_{m \neq n} \hat{v}_m(x_m^*) - \alpha_n \\
&= v_n(x_n^*) - p_n^* .
\end{aligned} \tag{1}$$

The last line is bidder  $n$ 's payoff from truthful reporting, so truthful reporting is always optimal. We omit the tedious but routine check that no other report is always optimal.

The last statement follows by construction of the mechanism. ■

### **3. Virtues of the VCG mechanism**

The VCG mechanism has several important virtues. The first is the dominant-strategy property, which reduces the costs of the auction by making it easier for bidders to determine their optimal bidding strategies and by eliminating bidders' incentives to spend resources learning about competitors' values or strategies. Such spending is pure waste from a social perspective, since it is not needed to identify the efficient allocation, yet it can be encouraged by auction formats in which each bidder's best strategy depends on its opponents' likely actions.

The dominant strategy property also has the apparent advantage of adding reliability to the efficiency prediction, because it means that the conclusion is not sensitive to assumptions about what bidders may know about each others' values and strategies. This is a distinctive virtue of the VCG mechanism. Theorems by Green and Laffont (1979) and by Holmstrom (1979) show that, under weak assumptions, the VCG mechanism is the unique direct reporting mechanism with dominant strategies, efficient outcomes, and zero payments by losing bidders. Here, we report a version of Holmstrom's theorem. To prove it, we need one extra assumption that has not been needed so far, namely, that the

set  $\mathcal{V}$  of possible value functions is *smoothly path connected*. This means that given any two functions in  $\mathcal{V}$ , there is a smoothly parameterized family of functions  $\{v(x,t)\}$  that lies wholly in  $\mathcal{V}$ , and connects the two functions. More precisely, for any two elements  $v(\bullet,0)$  and  $v(\bullet,1)$  in  $\mathcal{V}$ , there exists a path  $\{v(\bullet,t) | t \in [0,1]\}$  such that  $v$  is differentiable in its second argument and such that the derivative satisfies  $\int_0^1 \sup_x |v_2(x,t)| dt < \infty$ , where  $v_2 \equiv \partial v / \partial t$  here denotes the partial derivative of  $v$  with respect to the second argument.

Theorem 2. If the set of possible value functions  $\mathcal{V}$  is smoothly path connected and contains the zero function, then the unique direct revelation mechanism for which truthful reporting is a dominant strategy, the outcomes are always efficient, and there are no payments by or to losing bidders is the VCG mechanism.

Proof. Fix any values for the bidders besides bidder  $n$  and consider any mechanism satisfying the assumptions of the theorem. If  $n$  reports the zero function, then his VCG allocation is zero and his payoff is also zero. Suppose that  $n$  reports some value function  $v(\bullet,1)$  and let  $v(\bullet,0)$  be the zero function. By construction, a bidder with values of zero for every package is a losing bidder at any efficient allocation. Let  $\{v(\bullet,t) | t \in [0,1]\}$  be a smooth path of value functions, as defined above.

Denote the total-value-maximizing allocation when  $n$  reports  $v(\bullet,t)$  by  $x^*(t)$  and let  $V(t)$  denote  $n$ 's corresponding payoff in the VCG mechanism:

$V(t) = \max_s v(x_n^*(s), t) - p_n(s)$ . By the envelope theorem in integral form (Milgrom and Segal (2002)),  $V(1) - V(0) = \int_0^1 v_2(x^*(t), t) dt$ .

Let  $\hat{p}(t)$  be the payments made under any other direct revelation mechanism for which truthful reporting is a dominant strategy, the outcomes are always efficient, and there are no payments by or to losing bidders. Let  $\pi(t)$  denote  $n$ 's payoff in the alternate mechanism when his value function is  $v(\bullet, t)$ :  $\pi(t) = \max_s v(x_n^*(s), t) - \hat{p}_n(s)$ . By the envelope theorem in integral form,  $\pi(1) - \pi(0) = \int_0^1 v_2(x^*(t), t) dt = V(1) - V(0)$ .

Since there are no payments by or to losing bidders,  $\pi(0) = V(0) = 0$ , so  $v(x^*(1), 1) - \hat{p}_n(1) = \pi(1) = V(1) = v(x^*(1), 1) - p_n(1)$ . Hence,  $p_n(1) = \hat{p}_n(1)$ ; the payment rule must be the same as for the VCG mechanism. ■

Another virtue of the VCG mechanism is its scope of application. Theorems 1 and 2 above do not impose any restrictions on the bidders' possible rankings of different outcomes. The basic rules of the Vickrey auction can be further adapted if the auctioneer wishes to impose some extra constraints. For example, the government seller in a spectrum auction may wish to limit the concentration of spectrum ownership according to some measure. Or, the buyer in a procurement auction might want to limit its total purchases from first-time bidders or might want to ensure security by requiring that the total relevant capacity of its suppliers is at least 200% of the amount ordered. One can replace the constraint that  $\sum_m x_m \leq \bar{x}$  by any constraint of the form  $x \in X$  without affecting the preceding theory or arguments in any essential way.

A final virtue of the Vickrey auction is that its average revenues are not less than that from any other efficient mechanism, even when the notion of implementation is



expanded to include Bayesian equilibrium. A formal statement of this famous revenue equivalence theorem is given below.

Theorem 3. Consider a Bayesian model in which the support of the set of possible value functions,  $\mathcal{V}$ , is smoothly path connected and contains the zero function. Suppose the bidder value functions are independently drawn from  $\mathcal{V}$ . If, for some mechanism, the Bayesian-Nash equilibrium outcomes are always efficient and there are no payments by or to losing bidders, then the expected payment of each bidder  $n$ , conditional on his value function  $v_n \in \mathcal{V}$ , is the same as for the VCG mechanism. In particular, the seller's revenue is the same as for the VCG mechanism.<sup>2</sup>

#### **4. Weaknesses of the VCG mechanism**

Despite the attractiveness of the dominant-strategy property, the VCG mechanism also has several possible weaknesses:

- low (or zero) seller revenues;
- non-monotonicity of the seller's revenues in the set of bidders and the amounts bid;
- vulnerability to collusion by a coalition of losing bidders; and
- vulnerability to the use of multiple bidding identities by a single bidder.

It will be seen later in this chapter that a simple and intuitive condition on individual bidders characterizes whether these deficiencies are present. In economic environments where every bidder has *substitutes preferences*, the above-listed weaknesses will never occur. However, if there is but a single bidder whose preferences violate the substitutes

condition, then with an appropriate choice of values for the remaining bidders (even if the latter values are restricted to be additive), all of the above-listed weaknesses will be present.

In what follows, we will limit attention to auctions of multiple items, in which different bidders may want different numbers or different packages of items. One obvious reason for the disuse of the VCG mechanism for large-scale applications with diverse items is the same as for other “combinatorial” or “package” auctions: complexity in all aspects of its implementation. The chapters of this book on the winner determination problem give some insights into the problem facing the auctioneer. There are also important difficulties facing the bidder in such auctions.

Complexity, however, cannot be the whole explanation of the rarity of the VCG mechanism. Combinatorial auctions of other kinds have been adopted for a variety of procurement applications, from school milk programs in Chile to bus route services in London, England. Small-scale combinatorial auctions are technically feasible, so we are forced to conclude that VCG rules have not been employed even when they are feasible.

Analyses of the VCG mechanism often exclude any discussion of the auction revenues. This is an important omission. For private sellers, revenues are the primary concern. Even in the government-run spectrum auctions, in which priorities like allocational efficiency, promoting ownership by small businesses or women- or minority-owned businesses, rewarding innovation, and avoiding concentration of ownership are weighty considerations, one of the performance measures most emphasized by the public and politicians alike is the final auction revenue.<sup>3</sup>

Against this background, it is particularly troubling that the Vickrey auction revenues can be very low or zero, even when the items being sold are quite valuable and competition is ample. For example,<sup>4</sup> consider a hypothetical auction of two spectrum licenses to three bidders. Suppose that bidder 1 wants only the package of two licenses and is willing to pay \$2 billion, while bidders 2 and 3 are both willing to pay \$2 billion for a single license. The VCG mechanism assigns the licenses efficiently to bidders 2 and 3. The price paid by bidder 2 is the difference in the value of one license or two licenses to the remaining bidders. Since that difference is zero, the price is zero! The same conclusion applies symmetrically to bidder 3, so the total auction revenues are zero.

Notice that, in this example, if the government had sold the two licenses as an indivisible whole, then it would have had three bidders each willing to pay \$2 billion for the combined license. That is ample competition: an ascending auction would have been expected to lead to a price of \$2 billion. This revenue deficiency of the VCG mechanism is decisive to reject it for most practical applications.

Closely related to the revenue deficiency of the Vickrey auction is the non-monotonicity of seller revenues both in the set of bidders and in the amounts bid. In the preceding example, if the third bidder were absent or if its value for a license were reduced from \$2 billion to zero, then the seller's revenues in the VCG auction would increase from \$0 to \$2 billion. This non-monotonicity is not only a potential public relations problem but also creates loopholes and vulnerabilities that bidders can sometimes exploit.

One is the Vickrey design's vulnerability to collusion, even by losing bidders. This can be illustrated by a variant of the preceding example, in which bidder 1's values are

unchanged but bidders 2 and 3 have their values reduced so that each is willing to pay only \$0.5 billion to acquire a single license. The change in values alters the efficient allocation, making bidders 2 and 3 losers in the auction. However, notice that if the bidders each bid \$2 billion for their individual licenses, then the outcome will be the same as above: each will win one license at a price of zero! This shows that the VCG mechanism has the unusual vulnerability that even losing bidders may possess profitable joint deviations, facilitating collusion in the auction.

A closely related stratagem is a version of shill bidding—a bidder’s use of multiple identities in the auction—as discussed in chapter 7 of this book.<sup>5</sup> To construct this example, let us replace bidders 2 and 3 of the preceding example by a single combined bidder whose values are \$0.5 billion for a single license and \$1.0 billion for the pair of licenses. Again, the efficient allocation assigns the licenses to bidder 1, so the combined bidder is destined to lose. However, if the auctioneer cannot keep track of the bidders’ real identities, then the combined bidder could participate in the auction under two names—“bidder 2” and “bidder 3”—each of whom bids \$2 billion for a single license. As we have already seen, the result is that bidders 2 and 3 win at a price of zero, so the combined bidder wins both licenses at a total price of zero.

A related difficulty with the VCG mechanism arises even if all the bidders play their dominant strategies, with no collusion or multiple identities. Suppose again that bidders 2 and 3 each value a single license at \$2 billion, but that by combining operations they could increase the value of the package from \$4 billion to  $\$4+X$  billion. If all bidders bid their values as in the original Vickrey analysis, then the unmerged bidders 2 and 3 would pay zero and earn a total profit of \$4 billion. A merger by bidders 2 and 3 before the

auction would raise the price from zero to \$2 billion, so the firms would find it profitable to merge only if  $X > \$2$  billion. Thus, while the dominant strategy solution of the VCG mechanism provides the proper incentives for bidding with a fixed set of bidders, it distorts pre-auction decisions related to merging bidders.

Notice that, in each of these examples, bidder 1 has value only for the entire package. If we modified the examples to make the licenses *substitutes* for bidder 1, for example, by specifying that bidder 1 was willing to pay \$1 billion for each license rather than just \$2 billion for the pair, then all of our conclusions would change. The seller's total revenue in the first example would be \$2 billion, which is a *competitive* payment in the sense that the payoff outcome lies in the core.<sup>6</sup> This modification reverses all of the related examples, eliminating non-monotonicities in seller revenues, profitable joint deviations by losing bidders, the profitable use of multiple identities in bidding, and the bias against value-creating mergers.

The substitutes condition turns out to play a decisive role in characterizing the performance of the Vickrey auction. The formal analysis below develops that conclusion. We show that the kinds of problems described above cannot arise if all bidders are limited to having substitutes preferences. Moreover, if the possible bidder values include the additive values and if they are not restricted only to substitutes preferences, then there exist values profiles that exhibit all of the kinds of problems described above.

Before turning to that formal analysis, we illustrate some other, simpler drawbacks of the VCG auction design.

The Vickrey theory incorporates the very restrictive assumption that bidders' payoffs are quasi-linear. This requires that payoffs can be expressed as the value of the items

received minus the payment made. In particular, it requires that there is no effective budget limit to constrain the bidders and that the buyer, in a procurement auction, does not have any overall limit on its cost of procurement. Although we have no data on how frequently these assumptions are satisfied, it appears that failures may be common in practice.

It is easy to see that the dominant strategy property breaks down when bidders have limited budgets.<sup>7</sup> For example, consider an auction in which each bidder has a budget of \$1.2 billion to spend. Values are determined separately from budgets: they reflect the increment to the net present value of the bidder's profits if it acquires the specified spectrum licenses. Bidder A has values of \$1 billion for a single license or \$2 billion for the pair. Bidders B and C each want only one license. For bidder B, the value of a license is \$800 million. Bidder C's value is unknown to the other bidders, because it depends on whether C is able to acquire a particular substitute outside of the auction. Depending on the circumstances, bidder C may be willing to pay either \$1.1 billion or zero for a license. In a Vickrey auction, bidder A should win either two licenses or one, depending on the last bidder's decision. In either case, its total payment will be \$800 million, so its budget is always adequate to make its Vickrey payment. Yet, if A's budget limit constrains it to bid no more than \$1.2 billion for any package, then it has no dominant strategy. For suppose that bidders B and C adopt their dominant strategies, submitting bids equal to their values. Then, if bidder C bids zero, then A must bid less than \$400 million for a single license (and, say, \$1.2 billion for the package) to win both licenses and maximize its payoff. If instead bidder C bids \$1.1 billion, then A must bid more than \$800 million

for a single license to win a single license and maximize its payoff. Since A's best bid depends on C's bid, A has no dominant strategy.

Vickrey auctions are sometimes viewed as unfair because two bidders may pay different prices for identical allocations. To illustrate, suppose there are two bidders and two identical items. Bidder 1 bids 12 for a single item and 13 for the package, while bidder 2 bids 12 for a single item and 20 for the package. The result is that each bidder wins one item, but the prices they pay are 8 and 1, respectively, even though the items are identical and each has made the same bid for the item it wins.

Rothkopf, Teisberg and Kahn (1990) have argued that the VCG auction design presents a privacy preservation problem. Bidders may rationally be reluctant to report their true values, fearing that the information they reveal will later be used against them. For example, the public has sometimes been outraged when bidders for government assets are permitted to pay significantly less than their announced maximum prices in a Vickrey auction (McMillan (1994)), and such reactions can lead to cancellations, renegotiations, or at least negative publicity. Modern systems of encryption make it technically feasible to verify the auction outcome without revealing the actual bids, so the privacy concern may seem less fundamental than the others weaknesses reported above. Moreover, the widespread use of proxy bidders, in which bidders are asked to report their values or maximum bids, in electronic auctions conducted by Google (for ad placements), eBay, Amazon and others, establishes that privacy concerns are not always decisive.

Our entire discussion above has assumed the so-called "private values" model, in which each bidder's payoff depends solely on its own estimate of value and *not* on its opponents' estimates of value. This is appropriate, inasmuch as the classic theory of the

Vickrey auction and the VCG mechanism is developed for a private values model.

Without private values, they immediately lose their dominant-strategy property.<sup>8</sup>

Moreover, serious problems may arise when Vickrey auctions are applied in other frequently studied models. In the common value model of Milgrom (1981) and the “almost common value” model of Klemperer (1998), revenues can be nearly zero for a different reason than above, related to the extreme sensitivity of equilibria in the second-price auction to the winner’s curse. In the model of Bulow, Huang and Klemperer (1999), the allocational efficiency of the second price auction design discourages entry by bidders who are unlikely to be part of the efficient allocation. The paucity of entry can be another reason for very low prices.

## **5. VCG outcomes and the core**

Our discussion of the weaknesses of the VCG mechanism began with a three-bidder example in which the auction revenues were zero. Several of the subsequent weaknesses were built by varying this basic example. Since the zero revenues example was very special, a basic question will be: how low must revenue be before it is unacceptably low? We will adopt a competitive standard: the payoff outcome must lie in the core.

To justify the core as a competitive standard, suppose that two or more risk-neutral brokers compete to purchase the services of the auction participants. The broker’s profits are defined to be the value of the hired coalition minus the wages paid to those hired. It then follows that the competitive equilibrium price vectors of such an economy are the same as the payoff vectors in the core of the game. The reason is simple. First, the requirement of equilibrium that the brokers do not lose money is identical with the core requirement that the payoff allocation (or “imputation”) is feasible. Second, the



requirement of equilibrium that no broker can earn positive profits by any feasible strategy coincides exactly with the core requirement that no coalition can block the payoff allocation. The conditions imposed by equilibrium on the price vector are thus the same as those imposed by the core on payoff profiles.

To investigate the relationship between VCG outcomes and the core, we introduce some notation. We first define the coalitional game  $(L, w)$  that is associated with the trading model. The set of players is  $L = \{0, \dots, |L| - 1\}$ , with player 0 being the seller. The set of feasible allocations is  $X$ , for example,  $X = \{(x_l)_{l \in L-0} : x_l \geq 0 \text{ and } \sum_{l \in L-0} x_l \leq \bar{x}\}$ .

The coalitional value function is defined for coalitions  $S \subset L$  as follows:

$$w(S) = \begin{cases} \max_{x \in X} \sum_{l \in S} v_l(x_l), & \text{if } 0 \in S, \\ 0, & \text{if } 0 \notin S. \end{cases} \quad (2)$$

The value of a coalition is the maximum total value the players can create by trading among themselves. If the seller is not included in the coalition, that value is zero.

The core of a game with player set  $L$  and coalitional value function  $w(\bullet)$  is defined as follows:

$$\text{Core}(L, w) = \left\{ \pi : w(L) = \sum_{l \in L} \pi_l, w(S) \leq \sum_{l \in S} \pi_l \text{ for all } S \subset L \right\}.$$

Thus, the core is the set of profit allocations that are feasible for the coalition of the whole and unblocked by any coalition.

Let  $\bar{\pi}$  denote the *Vickrey payoff vector*:  $\bar{\pi}_l = w(L) - w(L-l)$  for bidders  $l \in L-0$  and  $\bar{\pi}_0 = w(L) - \sum_{l \in L-0} \bar{\pi}_l$  for the seller. The next several theorems and their proofs are taken from Ausubel and Milgrom (2002).

Theorem 4. A bidder's Vickrey payoff  $\bar{\pi}_l$  is  $l$ 's highest payoff over all points in the core. That is, for all  $l \in L-0$ :  $\bar{\pi}_l = \max \{ \pi_l \mid \pi \in \text{Core}(L, w) \}$ .

Proof. The payoff vector defined by  $\pi_0 = w(L-l)$ ,  $\pi_l = w(L) - w(L-l)$ , and  $\pi_j = 0$ , for  $j \neq 0, l$ , satisfies  $\pi \in \text{Core}(L, w)$ . Hence,  $\bar{\pi}_l \leq \max \{ \pi_l \mid \pi \in \text{Core}(L, w) \}$ .

Now suppose that  $\pi$  is a feasible payoff allocation with  $\pi_l > \bar{\pi}_l$  for some  $l \neq 0$ . Then  $\sum_{k \neq l} \pi_k = w(L) - \pi_l < w(L-l)$ , so coalition  $L-l$  blocks the allocation. Hence,  $\pi \notin \text{Core}(L, w)$ . ■

A payoff vector in the core is *bidder optimal* if there is no other core allocation that all bidders prefer. It is *bidder dominant* if it is the bidders' unanimously most preferred point in the core. More precisely, let  $\pi \in \text{Core}(L, w)$ . We say that  $\pi$  is bidder optimal if there is no  $\pi' \in \text{Core}(L, w)$  with  $\pi' \neq \pi$  and  $\pi'_l \geq \pi_l$  for every bidder  $l$ . We say that  $\pi$  is bidder dominant if every  $\pi' \in \text{Core}(L, w)$  satisfies  $\pi_l \geq \pi'_l$  for every bidder  $l$ .

Theorem 5. If the Vickrey payoff vector  $\bar{\pi}$  is in the core, then it is the bidder-dominant point in the core. If the Vickrey payoff vector  $\bar{\pi}$  is not in the core, then there is no bidder-dominant point in the core and the seller's Vickrey payoff is strictly less than the smallest of the seller's core payoffs.

Proof. By Theorem 4,  $\bar{\pi}_l \geq \pi_l$  for all  $\pi \in \text{Core}(L, w)$  and  $l \in L-0$ . Hence, if the Vickrey payoff  $\bar{\pi}$  is in the core, then it is bidder dominant.

Conversely, suppose that  $\bar{\pi} \notin \text{Core}(L, w)$  and consider any  $\hat{\pi} \in \text{Core}(L, w)$  and any  $j$  for whom  $\hat{\pi}_j \neq \bar{\pi}_j$ . By Theorem 4,  $\hat{\pi}_j < \bar{\pi}_j$ , so  $\hat{\pi}$  is not bidder dominant, and  $\hat{\pi}_l \leq \bar{\pi}_l$  for all  $l \in L - 0$ . So,  $\bar{\pi}_0 = w(L) - \sum_{l \in L-0} \bar{\pi}_l < w(L) - \sum_{l \in L-0} \hat{\pi}_l = \hat{\pi}_0$ . ■

An equivalent way of expressing Theorem 5 is that, if the Vickrey payoff is in the core, then it is the unique bidder-optimal point in the core. Otherwise, there is a multiplicity of bidder-optimal points in the core—none of which is bidder dominant.

We thus find that the seller's revenues from the VCG mechanism are lower than the competitive benchmark unless the core contains a bidder-dominant point. Only then is the Vickrey payoff in the core, and only then does the seller get competitive payoff for its goods. The next tasks are to investigate when the Vickrey payoff is in the core and to show that, as in the examples, the VCG mechanism's other weaknesses hinge on these same conditions.

The style of our analysis is to treat conditions that are robust to certain variations in the underlying model. One might motivate this by imagining that the auction designer does not know who will bid in the auction or precisely what values the bidders will have. The next two subsections deal with the uncertainties in sequence.

## **6. Conditions on the coalitional value function**

For this section, it is helpful to imagine that the seller knows the set of potential bidders  $L$  but does not know which of them will actually participate in the auction. The question we ask concerns conditions on the coalitional value function sufficient to ensure that the Vickrey outcome meets the competitive benchmark, that is, lies in the core,

regardless of which bidders decide to participate. This sort of robustness property is both practically valuable and analytically useful, as we will see below.

To explore the question described above, we introduce the concept of a “restricted Vickrey payoff vector”  $\bar{\pi}(S)$ , which applies to the cooperative game in which participation is restricted to the members of coalition  $S$ . Thus,  $\bar{\pi}_l(S) \equiv w(S) - w(S - l)$  for  $l \in S - 0$  and  $\bar{\pi}_0(S) \equiv w(S) - \sum_{l \in S - 0} \bar{\pi}_l(S)$ . We also introduce our main condition for this section:

**Definition.** The coalitional value function  $w$  is *bidder-submodular* if for all  $l \in L - 0$  and all coalitions  $S$  and  $S'$  satisfying  $0 \in S \subset S'$ ,  $w(S \cup \{l\}) - w(S) \geq w(S' \cup \{l\}) - w(S')$ .

**Theorem 6.** The following three statements are equivalent:

- (i) The coalitional value function  $w$  is bidder-submodular.
- (ii) For every coalition  $S$  that includes the seller, the (restricted) Vickrey payoff vector is in the core:  $\bar{\pi}(S) \in \text{Core}(S, w)$ .
- (iii) For every coalition  $S$  that includes the seller, there is a unique bidder-optimal core point in the restricted game and, moreover:

$$\text{Core}(S, w) = \{\pi_S \mid \sum_{l \in S} \pi_l = w(S), 0 \leq \pi_l \leq \bar{\pi}_l(S) \text{ for all } l \in S - 0\} \quad (3)$$

**Proof.** Define  $\Pi_S = \{\pi_S \mid \sum_{l \in S} \pi_l = w(S), 0 \leq \pi_l \leq \bar{\pi}_l(S) \text{ for all } l \in S - 0\}$ . Suppose that

(i) holds. It follows from Theorem 4 that  $\text{Core}(S, w) \subset \Pi_S$ . For the reverse inclusion, suppose  $\pi_S \in \Pi_S$  and that  $S = \{0, \dots, s\}$ , where  $s = |S| - 1$ . Let  $S'$  be a subcoalition of  $S$  including the seller, say,  $S' = \{0, \dots, k\}$ . We show that the blocking inequality associated with coalition  $S'$  is satisfied. This follows because:

$$\begin{aligned}
\sum_{l \in S'} \pi_l &= w(S) - \sum_{l=k+1}^s \pi_l \\
&\geq w(S) - \sum_{l=k+1}^s \bar{\pi}_l(S) \\
&= w(S) - \sum_{l=k+1}^s [w(S) - w(S-l)] \\
&\geq w(S) - \sum_{l=k+1}^s [w(\{0, \dots, l\}) - w(\{0, \dots, l-1\})] \\
&= w(S) - [w(S) - w(S')] \\
&= w(S')
\end{aligned} \tag{4}$$

The first step in (4) follows from the feasibility of  $\pi$ , the second from  $\pi \in \Pi$ , the third from the definition of  $\bar{\pi}$ , and the fourth from bidder-submodularity. Hence, (i) $\Rightarrow$ (iii).

It is immediate that (iii) $\Rightarrow$ (ii).

For (ii) $\Rightarrow$ (i), we show the contrapositive. Suppose (i) fails, that is,  $w$  is not bidder-submodular. Then, there exists a player  $i$  such that  $w(S) - w(S-i)$  is not weakly decreasing in  $S$ . Hence, there is a coalition  $S'$  including the seller and players  $i, j \in S' - 0$  such that  $w(S') - w(S'-i) > w(S'-j) - w(S'-ij)$ . So,  $\sum_{l \in S'-ij} \bar{\pi}_l(S') = w(S') - \bar{\pi}_i(S') - \bar{\pi}_j(S') = w(S'-i) + w(S'-j) - w(S') < w(S'-ij)$ . Thus, the payoff allocation  $\bar{\pi}_{S'}$  is blocked by coalition  $S'-ij$ , that is, (ii) also fails. ■

Theorem 6 gives a sharp and complete answer to our question: the seller can be sure that the VCG mechanism will satisfy the competitive benchmark regardless of which bidders participate exactly when the coalitional value function is bidder-submodular.

## 7. Conditions on the goods valuation functions

Coalitional values in an auction are not primitive objects. Rather, they are derived from individual bidders' package values using the formula of equation (2). Moreover, it is not usually reasonable to suppose that the auctioneer knows the coalitional value

function or the bidders' values from which they are derived. Also, a model of bidder values can include the case where some potential bidders might not participate by including the possibility that bidder values are zero. It thus seems worthwhile to investigate the conditions on individual preferences that determine whether VCG payoffs lie robustly in the core. As we have already hinted, the key to the analysis is the condition that goods are substitutes.

In discrete models, demands for goods are necessarily multi-valued for some prices, so we need to modify the standard definition of substitutes to accommodate that. Thus, we shall say that *goods are substitutes* if they are substitutes on the restricted price domain for which demand is single valued. In particular, goods are substitutes if the demand for each is nondecreasing in the prices of others on the relevant domain. Let  $v_l$  denote a buyer's value function for various packages and let  $u_l(p) \equiv \max_z \{v_l(z) - p \cdot z\}$  define the corresponding indirect utility function. Then, the substitutes condition has a convenient dual representation, as follows.

Theorem 7. Goods are substitutes for bidder  $l$  if and only if the indirect utility function  $u_l(\bullet)$  is submodular.

Proof. The following three conclusions follow directly from a version of the envelope theorem.<sup>9</sup> First, the indirect utility function  $u_l$  is absolutely continuous and differentiable almost everywhere in each good's price  $p_m$ . Second, the partial derivative  $\partial u_l / \partial p_m$  exists for precisely those price vectors  $p$  at which  $l$ 's demand for good  $m$  is single-valued. Third, at those prices  $p$ ,  $\partial u_l / \partial p_m = -x_{lm}(p)$ , where  $x_{lm}(p)$  is the quantity demanded of

good  $m$  at price vector  $p$  (that is,  $x_m(p)$  is the  $m^{\text{th}}$  coordinate of

$$x_l(p) = \arg \max_{\hat{x}_l} \{u_l(\hat{x}_l) - p \cdot \hat{x}_l\}.$$

By definition, the substitutes condition is satisfied if and only if  $x_m(p)$  is nondecreasing in each  $p_j$  for  $j \neq m$ . Thus, goods are substitutes if and only if  $\partial u_l / \partial p_m$  is nonincreasing in each  $p_j$  for  $j \neq m$ . This is equivalent to the conclusion that  $u_l$  is submodular. ■

It is a familiar fact that when the valuation  $v_l$  is concave, it can be recovered from its dual, the indirect utility function. Recovery is not possible for general non-concave valuations, because there do not generally exist prices that correspond to each quantity vector being demanded. In this problem, however, where each good is unique and priced separately, there are prices that support any demand vector. Consequently, as the next lemma verifies, the usual formula applies to recover the goods valuation function from the indirect utility function.

Lemma. Suppose  $z \in \{0,1\}^N$ , that is, all items are individually identified and priced. If the indirect utility function is  $u(p) = \max_z v(z) - p \cdot z$ , then the corresponding goods valuation function is:

$$v(z) = \min_{p \in \mathbb{R}_+^N} \{u(p) + p \cdot z\}. \quad (5)$$

Proof. For all  $p$  and  $z$ ,  $u(p) \geq v(z) - p \cdot z$ , so  $v(z) \leq u(p) + p \cdot z$ . Let  $B$  be a large number that exceeds the incremental value of any good to any package. Then,  $v(z) \leq \min_{p \in [0,B]^N} \{u(p) + p \cdot z\}$ . For any fixed  $z \in \{0,1\}^N$ , set  $p_m = 0$  if  $z_m = 1$  and  $p_m = B$  if

$z_m = 0$ . By inspection,  $u(p) = v(z) - p \cdot z$ . So,  $v(z) \geq \min_{p \in [0, B]^N} \{u(p) + p \cdot z\}$ . Hence,

$$v(z) = \min_{p \in [0, B]^N} \{u(p) + p \cdot z\}. \quad \blacksquare$$

Formula (5) is helpful for proving the following theorem.

**Theorem 8.** If goods are substitutes for all bidders, then the coalition value function is bidder-submodular.

The proof formalizes the following intuitive argument: First, if goods are substitutes for each member of the coalition, then they are substitutes also for the coalition itself. Second, when a member is added to a coalition, if that member is assigned some goods, that leaves fewer goods available for the remaining members, which affects them like raising the prices of the reassigned goods. Since the goods are substitutes, this “price increase” raises the demand for all other goods, that is, the marginal values of additional good rises. Thus, the marginal value of goods to a coalition is increasing in coalition size. The incremental value of an extra member to the coalition is the new member’s value of its package minus the coalition’s opportunity cost of that package. Since that opportunity cost is increasing in coalition size, the value of a member is decreasing in the coalition size, which is equivalent to the statement that the coalition value function is bidder-submodular.

**Proof.** For any coalition  $S$  that includes the seller, the value for package  $z$  is

$$v_S(z) = \max \left\{ \sum_{l \in S} v_l(x_l) \mid x \geq 0, \sum_{l \in S} x_l \leq z \right\}$$

and the corresponding indirect utility function is:



$$\begin{aligned}
u_S(p) &= \max_z \{v_S(z) - p \cdot z\} \\
&= \max_z \left\{ \max_{\{x \geq 0 \mid \sum x_l \leq z\}} \sum_{l \in S} v_l(x_l) - p \cdot z \right\} \\
&= \max_{x \geq 0} \max_{z \geq \sum x_l} \left\{ \sum_{l \in S} v_l(x_l) - p \cdot x_l - p \cdot \left( z - \sum x_l \right) \right\}, \\
&= \max_{x \geq 0} \left\{ \sum_{l \in S} v_l(x_l) - p \cdot x_l \right\} \\
&= \sum_{l \in S} \max_{x_l \geq 0} \{v_l(x_l) - p \cdot x_l\} \\
&= \sum_{l \in S} u_l(p).
\end{aligned} \tag{6}$$

As a sum of submodular functions,  $u_S(\bullet)$  is also submodular. (This corresponds to the step in the intuitive argument of establishing that if goods are substitutes for the individual bidders, then they are also substitutes for the coalition  $S$ ).

Consider a version of (5) that applies to the coalition  $S$ :

$$v_S(z) = \min_{p \in [0, B]^N} \{u_S(p) + p \cdot z\}. \tag{7}$$

where  $B$  is a large number, exceeding every marginal valuation, so that constraining prices to lie below  $B$  leaves the minimum value unchanged for all  $z$ . The objective function in (7) is continuous, antitone (“weakly decreasing”) and submodular in  $p$  and has weakly decreasing differences in  $(p, S)$ . Applying the Topkis monotonicity theorem, the set of minimizers has a maximum element  $p(S \mid z)$ , which is an isotone (“weakly increasing”) function of  $S$ . (This corresponds roughly to the step in the intuitive argument that the marginal value of any good is increasing in coalition size. However, the next step is to show that the price  $p(S \mid z)$  corresponds to the marginal value for goods  $m$  such that  $z_m = 1$ .)

Fix any good  $m$ . If  $z_m = 0$ , then, by inspection,  $p_m(S \mid z) = B$ . Next, suppose that  $z_m = 1$ . Let  $\varepsilon > 0$  and  $p'_\varepsilon = p(S \mid z) + \varepsilon 1_m$ . By definition of  $p(S \mid z)$ , demand for good  $m$

at price vector  $p'_\varepsilon$  is zero. Since  $p_m(S | z) = B$  for goods  $j$  for which  $z_j = 0$ , demand for these goods is still zero at price vector  $p'_\varepsilon$ . By the condition of substitutes, demand for the remaining goods is undiminished. Hence,  $z - 1_m \in \arg \max_{z'} v_S(z') - p'_\varepsilon \cdot z'$  for all  $\varepsilon > 0$ .

By the theorem of the maximum, the same must hold for  $\varepsilon = 0$ , that is,

$$z - 1_m \in \arg \max_{z'} v_S(z') - p(S | z) \cdot z'. \text{ So}$$

$$v_S(z - 1_m) - p(S | z) \cdot (z - 1_m) = v_S(z) - p(S | z) \cdot z, \text{ and hence } p_m(S | z) = v_S(z) - v_S(z - 1_m).$$

Let  $z^n = (1, \dots, 1, 0, \dots, 0)$  denote a vector with  $n$  initial 1's. Without loss of generality, we may reorder the items so that  $z = z^m$  for some  $m$ . Then,

$$v_S(z^N) - v_S(z^m) = \sum_{j=m+1}^n (v_S(z^j) - v_S(z^{j-1})) = \sum_{j=m+1}^n p_j(S | z^j). \quad (8)$$

We have already established that  $p_j(S | z)$  is isotone in  $S$  for all  $z$ , so by (8),

$$v_S(z^N) - v_S(z^m) \text{ is isotone in } S \text{ as well.}$$

Notice that  $w(S \cup \{l\}) - w(S) = \max_z v_S(z) + v_l(M - z) - v_S(M)$ . By the preceding paragraph, the right-hand expression is a maximum of isotone functions of  $S$ , so  $w(S \cup \{l\}) - w(S)$  is itself isotone in  $S$ . Hence,  $w$  is bidder-submodular. ■

Thus, we have established that the substitutes condition for goods values is sufficient to conclude that the coalitional value function is bidder-submodular. If the possible bidder values for goods include all the additive functions, then the same condition is also necessary. To state this result clearly, we introduce three sets of value functions for goods. Let  $\mathcal{V}$  denote the set of value functions that are possible for bidders in the auction,

$\mathcal{V}_{add}$  the set of additive value functions, and  $\mathcal{V}_{sub}$  the set of value functions for which the goods are substitutes.

Theorem 9. Suppose that there are at least four possible bidders. Further suppose that there is a single unit of each kind and that  $\mathcal{V}_{add} \subset \mathcal{V}$ . Then the following three conditions are equivalent:

- (1)  $\mathcal{V} \subset \mathcal{V}_{sub}$
- (2) For every profile of bidder value functions drawn for each bidder from  $\mathcal{V}$ , the coalitional value function is bidder-submodular.
- (3) For every profile of bidder value functions drawn for each bidder from  $\mathcal{V}$ ,  $\bar{\pi} \in Core(L, w)$ .

Proof. By Theorems 8 and 6, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It remains to show that (3) $\Rightarrow$ (1).

Suppose that the substitutes condition fails for some  $v_1 \in \mathcal{V}$ , which we may take to be the value function of bidder 1. Then, there exist two goods,  $m$  and  $n$ , and a price vector,  $p$ , with  $p_n, p_m > 0$ , such that for all  $\hat{p}_m$  such that  $0 \leq \hat{p}_m < p_m$ , there is a unique maximizer  $x'$  of  $v_1(x) - (\hat{p}_m, p_{-m}) \cdot x$  satisfying  $x'_n = x'_m = 1$ , and for all  $\hat{p}_m > p_m$ , there is a unique maximizer  $x''$  satisfying  $x''_n = x''_m = 0$ .<sup>10</sup> By Berge's theorem, it follows that at the price vector  $p$ , both  $x'$  and  $x''$  are optimal. Moreover, any alternative bundle,  $x'''$ , with the property that  $x'''_m = 0$  and  $x'''_n = 1$  is strictly suboptimal at price vector  $p$ .<sup>11</sup>

Since  $\mathcal{V}_{add} \subset \mathcal{V}$ , we may take buyers 2, 3 and 4 to have additive value functions as follows:  $v_2(x) = \sum_{k \neq n, m} p_k x_k$ ;  $v_3(x) = p_m x_m + p_n x_n$ ; and  $v_4(x) = \hat{p}_m x_m$  where  $\hat{p}_m > p_m$ .

Since  $x'$  is optimal for buyer 1 at price vector  $p$  above,  $w(0123) = w(012)$ . Since  $x''$  is the unique optimum for buyer 1 at price vector  $(p_{-m}, \hat{p}_m)$  and since  $p_n > 0$ ,

$w(01234) > w(0124)$ . At the Vickrey payoffs,  $\bar{\pi}_0 + \bar{\pi}_1 + \bar{\pi}_2 = w(01234) - (\bar{\pi}_3 + \bar{\pi}_4) =$   
 $w(01234) - ([w(01234) - w(0124)] + [w(01234) - w(0123)]) = w(0123) + w(0124) -$   
 $w(01234) < w(0123) = w(012)$ . So, coalition 012 blocks the Vickrey payoff allocation and  
 $\bar{\pi} \notin \text{Core}(L, w)$ . ■

One joint implication of Theorems 8 and 9 is that if each bidder  $n$  is permitted to select its report *independently* from a set  $\mathcal{V}_n$  of possible reports of value functions—and if all reports of additive values are allowed—then the substitutes condition is both necessary and sufficient to assure that the coalitional value function is bidder-submodular (and that the Vickrey payoff allocation is contained in the core). While some other authors have observed that there exist profiles of reports that jointly satisfy bidder-submodularity but fail the substitutes condition for individual bidders, this observation can be misleading. The force of our theorems is that if the information available to the auction designer comes in the form of restrictions on the individual bidders' possible values and if the sets of possible values are “plausibly large,” then the auctioneer cannot be sure that the Vickrey outcome lies in the core. The difficulty of ruling out very bad outcomes when goods may not be substitutes is enough to explain the reluctance of many to use the VCG auction design.

Theorem 9 is closely related to theorems about the existence of competitive equilibrium goods prices in models like this one. Indeed, Milgrom (2000) shows that if  $\mathcal{V}_{add} \subset \mathcal{V}$ , then a competitive equilibrium exists for every profile of bidder value functions drawn from  $\mathcal{V}$  if and only if  $\mathcal{V} \subset \mathcal{V}_{sub}$ . Thus, possible failures of the substitutes

condition are problematic for traditional market mechanisms as well as for the Vickrey mechanism.

In presenting our examples, we claimed that failures of the substitutes condition are also closely connected to possibilities for manipulation in the Vickrey auction, including collusion by losers and the profitable use of bidding with multiple identities. We now show how these possibilities are connected to a failure of monotonicity of revenues in the set of bidders.

We define an auction to exhibit *bidder monotonicity* if adding another bidder always (weakly) reduces existing bidders' equilibrium profits and (weakly) increases the seller's equilibrium revenues. Bidder monotonicity formalizes the familiar property of ordinary single-item private-values auctions that increasing bidder participation can only benefit the seller. The next theorem shows that the substitutes condition is sufficient for bidder monotonicity in the Vickrey auction, and for loser collusion and bidding with multiple identities to be unprofitable.<sup>12</sup> Moreover, the substitutes condition is also necessary for these conclusions if the set of bidder values is otherwise sufficiently inclusive.

Theorem 10. Suppose that there is a single unit of each kind and that  $\mathcal{V}_{add} \subset \mathcal{V}$ . Then the following four conditions are equivalent:<sup>13</sup>

- (1)  $\mathcal{V} \subset \mathcal{V}_{sub}$ .
- (2) For every profile of bidder value functions drawn from  $\mathcal{V}$ , the VCG mechanism exhibits bidder monotonicity.
- (3) For every profile of bidder value functions drawn from  $\mathcal{V}$ , any bidding with multiple identities is unprofitable in the Vickrey auction.
- (4) For every profile of bidder value functions drawn from  $\mathcal{V}$ , any joint deviation by losing bidders is unprofitable in the Vickrey auction.

Proof. (1) $\Rightarrow$ (2): Let  $S, S'$  ( $0 \in S \subset S' \subset L$ ) be any nested sets of players that include the seller. As before,  $\bar{\pi}(S)$  and  $\bar{\pi}(S')$  denote the associated vectors of Vickrey payoffs.

By Theorem 8, if every bidder has substitutes preferences, then the coalitional value function is bidder-submodular. Hence, for every bidder  $l \in S - 0$ , we have:

$\bar{\pi}_l(S') = w(S') - w(S' - l) \leq w(S) - w(S - l) = \bar{\pi}_l(S)$ . The bidder-submodularity of the coalitional value function also implies that:  $\sum_{l \in S' - S} \bar{\pi}_l(S') \leq w(S') - w(S)$  (see Ausubel

and Milgrom, 2002, footnote 35). Summing these inequalities, we conclude:

$$\bar{\pi}_0(S') = w(S') - \sum_{l \in S' - 0} \bar{\pi}_l(S') \geq w(S) - \sum_{l \in S - 0} \bar{\pi}_l(S) = \bar{\pi}_0(S), \text{ as required.}$$

(1) $\Rightarrow$ (3): Let  $v$  be any profile of reports, let  $x$  be the associated VCG allocation, and let  $S \subset L - 0$  be any coalition of bidders. Suppose that the reports of every bidder  $l \in L - S$  satisfy the substitutes condition. As shown in the proof of Theorem 8,  $v_{L-S}(\bullet)$  is also submodular, because if goods are substitutes for the individual bidders, then they are also substitutes for the coalition  $L - S$ . Now consider any bidder  $k \in S$ . Recall that  $\bar{x}$  denotes the vector of goods being offered. Then  $v_L(\bar{x}) = v_{L-S}(x_{L-S}) + v_{S-k}(x_{S-k}) + v_k(x_k)$  and  $v_{L-k}(\bar{x}) \geq v_{L-S}(x_{L-S} + x_k) + v_{S-k}(x_{S-k})$ , since it is feasible to reassign  $x_k$  to coalition  $L - S$  in bidder  $k$ 's absence. Therefore:

$$\bar{\pi}_k(L) = v_L(\bar{x}) - v_{L-k}(\bar{x}) \leq v_k(x_k) - [v_{L-S}(x_{L-S} + x_k) - v_{L-S}(x_{L-S})].$$

Since  $\bar{\pi}_k(L) = v_k(x_k) - p_k$ , where  $p_k$  denotes bidder  $k$ 's payment in the VCG mechanism, this implies:

$$p_k \geq v_{L-S}(x_{L-S} + x_k) - v_{L-S}(x_{L-S}) \geq v_{L-S}(\bar{x}) - v_{L-S}(\bar{x} - x_k), \quad (9)$$

where the second inequality follows from the submodularity of  $v_{L-S}(\bullet)$ . (The intuition for inequality (9) is that the price paid by bidder  $k$  for  $x_k$  must be at least the opportunity cost of denying these goods to coalition  $L-S$ .)

Now suppose that a given bidder utilizes “skills” (i.e., multiple identities). Let  $S = \{1, \dots, |S|\}$  denote the coalition of these skills. Then,

$$\begin{aligned} \sum_{k=1}^{|S|} p_k &\geq \sum_{k=1}^{|S|} \left[ v_{L-S} \left( \bar{x} - \sum_{j \in S-k} x_j \right) - v_{L-S} \left( \bar{x} - \sum_{j \in S} x_j \right) \right] \\ &\geq \sum_{k=1}^{|S|} \left[ v_{L-S} \left( \bar{x} - \sum_{j=1}^{k-1} x_j \right) - v_{L-S} \left( \bar{x} - \sum_{j=1}^k x_j \right) \right] \\ &= v_{L-S}(\bar{x}) - v_{L-S} \left( \bar{x} - \sum_{j=1}^{|S|} x_j \right). \end{aligned}$$

The first inequality follows from (9) and the second from the submodularity of  $v_{L-S}$ . The sum telescopes to the last term, which is the price that the bidder utilizing skills would pay to acquire the same allocation in the VCG mechanism without skills. Hence, the use of skills is unprofitable.

(1) $\Rightarrow$ (4): Let  $v$  be the maximum value function for the coalition of winning bidders.

Since goods are substitutes for the winning bidders,  $v$  is submodular. Suppose a coalition  $S$  of losing bidders deviates and acquires the bundles  $(\hat{x}_l)_{l \in S}$ . Using inequality (9), the

Vickrey price paid by any losing bidder  $l$  to acquire its bundle is at least

$$v(\bar{x} - \sum_{k \in S-l} \hat{x}_k) - v(\bar{x} - \sum_{k \in S} \hat{x}_k) \geq v(\bar{x}) - v(\bar{x} - \hat{x}_l) \geq v_l(\hat{x}_l).$$

The first inequality follows from (9); the second follows because the middle expression is the Vickrey price for a lone deviator  $l$  for bundle  $\hat{x}_l$ , which is less than its value (because  $l$  is a losing bidder).

We conclude that no coalition of losing bidders has a profitable joint deviation.

(3) $\Rightarrow$ (1): Suppose that the substitutes condition fails for some value function  $v_1 \in V$ , which we may take to be the value function of buyer 1. Then exactly as in the proof of Theorem 9, there exist two goods,  $m$  and  $n$ , a price vector,  $p$ , and two bundles  $x'$  and  $x''$ , such that at price vector  $p$ ,  $\{x', x''\} = \arg \max \{v_1(x) - p \cdot x\}$ , where  $x'_n = x'_m = 1$  and  $x''_n = x''_m = 0$ . Let  $\Delta = v(x''_{-n}, 1) - v(x'')$  be the incremental value of good  $n$  to buyer 1 at  $x''$ . Following the proof of Theorem 9, we may take buyers 2 and 3 to have the additive value functions of  $v_2(x) = \sum_{k \neq n, m} p_k x_k$  and  $v_3(x) = p_m x_m + p_n x_n$ . Observe that, in the sincere bidding equilibrium of the Vickrey auction, buyer 3 receives a payoff of zero. However, suppose that buyer 3 can enter an additional bid,  $v_4(x) = (p_m + p_n - \Delta)x_m$ , under the false name of buyer 4. With this bid, the shill buyer 4 wins good  $m$  for payment of  $p_m$ . Also, buyer 3 then wins good  $n$  for payment of  $\Delta < p_n$ , so the shill bidding is profitable for buyer 3.

(2) $\Rightarrow$ (1): The construction in the preceding paragraph also provides a violation of bidder monotonicity. Adding buyer 4, now a real bidder, reduces the seller's revenues.

(4) $\Rightarrow$ (1): The preceding construction also applies if buyer 4 is a real bidder but has value function  $v_4(x) = p_m x_m$ . In that case, it illustrates profitable collusion among losing bidders. ■

## 8. Conclusion

Although the VCG mechanism has virtues that are important for applications, it also suffers from serious weaknesses that limit its usefulness.



The most obvious disadvantages of the design are the complexity of the problem it poses to bidders, the reluctance of bidders to reveal their values, and the strategic issues posed by budget constraints. Complexity problems are universal in auctions where bidders are required to enumerate values associated with all subsets or the auctioneer must solve a large integer programming problem. Bidder reluctance to reveal values is most significant when that information might leak out and adversely affect other decisions or negotiations. Budget constraints are serious. If budget limits applied to bids, then, as we showed, they can destroy the dominant strategy property even when there is no chance that the price charged will exceed the bidder's budget.

In the case of homogeneous goods and nonincreasing marginal values, these three disadvantages of a Vickrey auction can be avoided by using the efficient ascending auction of Ausubel (1997, 2002), in which the auctioneer iteratively announces a price and bidders respond with quantities. Items are awarded at the current price whenever they are "clinched," in the sense that the aggregate demand of a bidder's opponents becomes less than the available supply. The price continues to be incremented until the market clears.

With private values, the Ausubel auction yields the same outcome as the multi-unit Vickrey auction: truthful reporting is a subgame perfect equilibrium of the game; and in a discrete formulation with incomplete information, it is the unique outcome of iterated elimination of weakly dominated strategies. In the Ausubel auction, privacy is preserved, because bidders never report their demands at any prices exceeding the clearing price. And the Ausubel auction performs better in the face of budget constraints, because

bidders have the opportunity to reduce their demands to keep within their budgets as the bidding evolves.

When the items being sold may not be substitutes,<sup>14</sup> the VCG mechanism suffers from additional problems that are decisive for most practical applications. It can generate very low revenues—so low that the outcome is not in the core. Revenues can fall when additional bidders are introduced and or when bidders values are increased. Collusion is facilitated: even losing bidders can have profitable joint deviations. Also, bidders can sometimes profitably pose as multiple bidders, using pseudonyms to bid and increase their profits. If bidders can have any additive valuation functions, then these several additional problems can be excluded only when the goods are substitutes for the bidders. Thus, the problems are serious in the very case of complementary goods for which package bidding is believed to have its greatest potential payoff. Moreover, even when goods are substitutes, the VCG mechanism still has other imperfections. As we showed by example, it can discourage value-creating mergers among bidders.

For all of these reasons, it is useful to think of the VCG theory as a lovely and elegant reference point—but not as a likely real-world auction design. Better, more practical procedures are needed. We offer our own candidate designs in Ausubel and Milgrom (2002) and in chapters 3 and 5 of this book.

## References

- Ausubel, Lawrence M. (1997), "An Efficient Ascending-Bid Auction for Multiple Objects," forthcoming, *American Economic Review*.
- Ausubel, Lawrence M. (1999), "A Mechanism Generalizing the Vickrey Auction," Working Paper, University of Maryland.
- Ausubel, Lawrence M. (2002), "An Efficient Dynamic Auction for Heterogeneous Commodities," Working Paper, University of Maryland.
- Ausubel, Lawrence M. and Peter Cramton (1999), "The Optimality of Being Efficient," Working Paper, University of Maryland.
- Ausubel, Lawrence M. and Paul Milgrom (2002), "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, Vol. 1: No. 1, Article 1.  
<http://www.bepress.com/bejte/frontiers/vol1/iss1/art1>
- Bulow, Jeremy, Ming Huang and Paul Klemperer (1999), "Toeholds and Takeovers," *Journal of Political Economy*, **107**(3): 427-454.
- Che, Yeon-Koo and Ian Gale (1998), "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, **65**: 1-21.
- Clarke, E.H. (1971), "Multipart Pricing of Public Goods," *Public Choice*, **XI**, 17-33.
- Green, Jerry and Jean-Jacques Laffont (1979), *Incentives in Public Decision Making*, North Holland: Amsterdam.
- Groves, Theodore (1973), "Incentives in Teams," *Econometrica*, **41**: 617-31.
- Holmstrom, Bengt (1979), "Groves Schemes on Restricted Domains," *Econometrica*, **47**: 1137-1144.

- Klemperer, Paul (1998), "Auctions with Almost Common Values: The Wallet Game and Its Applications," *European Economic Review*, **42**: 757-769.
- Krishna, Vijay and Motty Perry (1997), "Efficient Mechanism Design," Working Paper, Penn State University.
- McMillan, John (1994), "Selling Spectrum Rights," *Journal of Economics Perspectives*, **8**: 145-162.
- Milgrom, Paul R. (1981), "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, **49**(4): 921-943.
- Milgrom, Paul (2000), "Putting Auctions Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, **108**(2): 245-272.
- Milgrom, Paul (2004), *Putting Auction Theory to Work*, Cambridge: Cambridge University Press.
- Milgrom, Paul and Ilya Segal (2002), "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, **70**(2): 583-601.
- Rothkopf, Michael, Thomas Teisberg and Edward Kahn (1990), "Why Are Vickrey Auctions Rare?" *Journal of Political Economy*, **98**: 94-109.
- Sakurai, Yuko, Makoto Yokoo and Shigeo Matsubara (1999), "A Limitation of the Generalized Vickrey Auction in Electronic Commerce: Robustness Against False-Name Bids," *Proceedings of the Sixteenth National Conference on Artificial Intelligence*: 86-92.
- Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, **16**: 8-37.

Williams, Steven R. (1999), “A Characterization of Efficient, Bayesian Incentive Compatible Mechanisms,” *Economic Theory*, 14(1), 155-180.

Yokoo, Makoto, Yuko Sakurai and Shigeo Matsubara (2000), “The Effect of False-Name Declarations in Mechanism Design: Towards Collective Decision Making on the Internet,” *Proceedings of the Twentieth International Conference on Distributed Computing Systems*: 146-153.

Yokoo, Makoto, Yuko Sakurai and Shigeo Matsubara (2004), “The Effect of False-Name Bids in Combinatorial Auctions: New Fraud in Internet Auctions,” *Games and Economic Behavior*, 46(1), 174-188.

---

<sup>1</sup> The Clarke-Groves design was introduced in Clarke (1971) and Groves (1973).

<sup>2</sup> The proof of Theorem 3 is similar to that for Theorem 2, relying on the envelope theorem to determine the expected payoff of each type of each bidder. Williams (1999) proves more generally that optimization over the class of dominant-strategy mechanisms yields the same objective as optimization over the (larger) class of Bayesian mechanisms. Krishna and Perry (1997) show that the VCG mechanism maximizes the seller’s revenues over the class of efficient mechanisms satisfying incentive compatibility and individual rationality.

<sup>3</sup> However, Ausubel and Cramton (1999) argue that in the extreme case of an auction followed by a resale market in which all available gains from trade are realized, a focus solely on efficiency is appropriate.

<sup>4</sup> This and all subsequent examples are drawn from Ausubel and Milgrom (2002).

<sup>5</sup> See Sakurai, Yokoo and Matsubara (1999) and Yokoo, Sakurai and Matsubara (2000, 2004) for the original treatments of what they have called “false name bidding”.

---

<sup>6</sup> In the unmodified example, the Vickrey outcome was not in the core. The coalition consisting of the seller and bidder 1 could “block” the Vickrey outcome since, by themselves, they earn a coalition payoff of \$2 billion, but the Vickrey outcome gave them only a payoff of zero.

<sup>7</sup> Che and Gale (1998) analyze revenue differences among first-price and second-price auctions in the presence of budget constraints.

<sup>8</sup> A generalization of the multi-unit Vickrey auction to environments where bidders receive one-dimensional signals, and each bidder’s values may depend on its opponents’ signals, is described in Ausubel (1999). With bidders’ values depending on opponents’ signals, the efficient mechanism cannot possess the dominant-strategy property, but it retains the somewhat weaker property that truthful reporting is an *ex post* Nash equilibrium.

<sup>9</sup> See Milgrom and Segal (2002), Corollary 4.

<sup>10</sup> The detailed construction begins by noting that, if the substitutes condition fails, then for any  $\varepsilon > 0$ , there exist two goods,  $m$  and  $n$ , and a price vector,  $(\bar{p}_m, p_{-m})$ , such that: (i) buyer 1 has unique demands at price vector  $(\bar{p}_m, p_{-m})$  and  $x_{1n}(\bar{p}_m, p_{-m}) = 1$ ; and (ii) buyer 1 has unique demands at price vector  $(\bar{p}_m + \varepsilon, p_{-m})$  and  $x_{1n}(\bar{p}_m + \varepsilon, p_{-m}) = 0$ . Hence, there exists  $p_m \in (\bar{p}_m, \bar{p}_m + \varepsilon)$  at which the demand for good  $n$  changes from 1 to 0. Furthermore,  $x_{1m}(\bar{p}_m, p_{-m}) \neq x_{1m}(\bar{p}_m + \varepsilon, p_{-m})$ , for otherwise (given the quasilinear preferences), the change in the price of good  $m$  would have no effect on buyer 1’s demand for good  $n$ .

<sup>11</sup> Otherwise, by inspection,  $x^m$  would also be optimal at any price vector  $(\hat{p}_m, p_{-m})$  where  $\hat{p}_m > p_m$ , contradicting the uniqueness of the optimum at such prices.

<sup>12</sup> Makoto Yokoo, Yuko Sakurai and Shigeo Matsubara show in a series of papers (Sakurai, Yokoo and Matsubara 1999, Yokoo, Sakurai and Matsubara 2000, 2004, and chapter 7 of this volume) that false-name bidding is unprofitable under increasingly general conditions. The most general condition requires that the set of possible bidder valuations be restricted to ones for which the coalitional value function is bidder submodular. Theorem 9 of this chapter provides a condition on the primitive preferences that implies this property of the coalitional value function.

---

<sup>13</sup> For models starting with a fixed number of bidders  $N$ , the proof establishes that  $(1) \Rightarrow (2)$ ,  $(1) \Rightarrow (3)$ ,  $(1) \Rightarrow (4)$ ,  $(2) \Rightarrow (1)$  and  $(3) \Rightarrow (1)$ , provided  $N \geq 3$ , and  $(4) \Rightarrow (1)$ , provided  $N \geq 4$ .

<sup>14</sup> Note that, in the case of homogeneous goods, the substitutes condition reduces to the condition of nonincreasing marginal values that was considered in the previous two paragraphs.